

# Axial perturbations of general spherically symmetric spacetimes

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## Abstract

The aim of this paper is to present a governing equation for first order axial metric perturbations of general, not necessarily static, spherically symmetric spacetimes. Under the non-restrictive assumption of axisymmetric perturbations, the governing equation is shown to be a two-dimensional wave equation where the wave function serves as a twist potential for the axisymmetry generating Killing vector. This wave equation can be written in a form which is formally a very simple generalization of the Regge-Wheeler equation governing the axial perturbations of a Schwarzschild black hole, but in general the equation is accompanied by a source term related to matter perturbations. The case of a viscous fluid is studied in particular detail.

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## 1 Introduction

To consider perturbations of various physical fields around a spherically symmetric background is natural for several reasons. From the mathematical point of view, the symmetries of the background simplify the equations considerably and make them tractable for taking the analytical treatment further and for carrying out the numerical calculations needed without going far beyond standard methods. From the physical point of view, many systems (in this paper, the ones in mind are mainly astrophysical) owe their main structure to their monopole moment, and may consequently be regarded as perturbations of spherically symmetric systems to a good approximation. Moreover, as far as e.g. gravitational and electromagnetic fields are concerned, being non-radiative when restricted to be spherically symmetric, it is of course highly interesting to see what types of radiation are allowed for their more general neighbours.

The discussion of this paper will be restricted to integer spin fields, satisfying linear differential equations defined on a spherically symmetric background. All such fields are known to split naturally into two parts that are not coupled by the equations of motion, namely an odd parity (or axial) part and an even parity (or polar) part. A standard way of separating out the angular dependence from the equations of motion for such fields is to make use of scalar, vector and tensor spherical harmonics (cf. [1]), which all can be

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defined in terms of the scalar ones, i.e.  $Y_l^m(\theta, \phi)$ , and derivatives thereof. As the final form of the equations are known not to depend on the separation constant  $m$ , there is no mathematical restriction of setting  $m$  to zero from the outset, corresponding to only considering axisymmetric fields. This was for instance the approach taken by Chandrasekhar in his extensive treatment of gravitational perturbations of Schwarzschild and Reisser-Nordström black holes [2], which summed up and added insight to the original works of Regge and Wheeler [3] and Zerilli [4]. Later, the same approach was taken by Chandrasekhar and Ferrari [5] when generalizing to static spherically symmetric perfect fluid stellar models. The main focus of this paper is to use the axisymmetric approach to present a general treatment of the axial gravitational perturbations, valid for all spherically symmetric backgrounds and with no restriction on the types of matter perturbations. Only after this work was completed, this author became aware of the work of Gerlach and Sengupta [6] who presented such a treatment for both polar and axial perturbations, using a  $2+2$  splitting scheme. However, new approaches and results are given in the present paper. Firstly, the method used to derive the final form of the perturbation equations, being a rather straightforward application of the general covariant form of the perturbed Einstein equations, differs from the one used in [6]. Here, the axial perturbation equations are first covariantly reduced to a Maxwell-like equation on the background geometry, valid not only for spherically symmetric backgrounds, but for any background with a non-null and twist-free Killing vector that is perturbed in such a way that the Killing vector obtains a twist to first order. The Killing vector is in this work identified with the axisymmetry generator of the neighbour to the spherically symmetric background, but it could e.g. just as well be identified with the time-like Killing vector of a stationary neighbour to a static background, meaning that the Maxwell-like equation can also be used in other contexts. Secondly, the further reduction of perturbation equations is shown to always lead to a governing wave equation of the Regge-Wheeler type  $(-\partial_t^2 + \partial_x^2 - V)\psi = S$ , with  $t$  and  $x$  being harmonically conjugate coordinates on the 2-space orthogonal to the  $SO(3)$  group orbits,  $V$  being a potential only depending on the background geometry and  $S$  being a source term directly related to the perturbation of the stress-energy tensor. It is found that the potential  $V$  can be written in a form which is a strikingly simple generalization of the form found by Chandrasekhar and Ferrari for the case of static perfect fluids: the energy density and pressure combination  $\rho - p$  simply has to be replaced by  $\rho - p_\perp$ , where  $p_\perp$  is the radial pressure of a radially moving observer who measures  $\rho$  as the energy density. It is also made clear that the wave function  $\psi$  simply corresponds to a two-dimensional twist potential of the axisymmetry generator, given of course that the perturbations are axisymmetric. Thirdly, for static perfect fluid backgrounds it is known that the source term  $S$  has to vanish unless the perturbation is stationary, in which case it corresponds to adding a small rotation to the fluid ball. Since it thus would be interesting to see what kinds of source terms are allowed when considering more general fluids, we discuss in detail a fluid with viscosity and find that  $S$  is in general non-vanishing, which corresponds to an interaction between shear and gravitational oscillations.

In addition to the treatment of axial gravitational perturbations, we also consider Maxwell and Klein-Gordon test fields on a general spherically symmetric background. Also in these cases the equations of motion are found to be reducible to Regge-Wheeler type wave equations of the form given above. In fact, it is found that the potential  $V$  for the three types of fields can be written in the common form  $V = V_s + V_l$ , where  $V_s$  and  $V_l$  depends quadratically on the spin  $s$  and the multipole moment  $l$ , respectively. The  $V_l$  part can notably be identified with the centrifugal potential for null geodesics. Such a splitting of the wave potential into a spin part and a centrifugal (or angular momentum) part has earlier been discovered for Schwarzschild black holes [7], but to the author's knowledge it has not been shown to generalize to the general spherically symmetric case.

## 2 Spacetimes admitting a nearly twist-free Killing vector

If a spacetime  $(M, g_{ab})$  admits a Killing vector  $\eta^a$  which is close to being twist-free, i.e. hypersurface orthogonal, it is natural to treat it as a perturbation of a spacetime admitting a Killing vector which is *exactly* hypersurface orthogonal. We shall therefore consider a one-parameter family of spacetimes with an associated family of Killing vectors  $\eta^a$ . The metrics of these spacetimes will be written in the form

$$g_{ab} = \perp_{ab} + F\mu_a\mu_b \quad (1)$$

where

$$F = \eta^a \eta_a \quad \mu_a = F^{-1} \eta_a \quad (2)$$

and  $\perp_a{}^b$  is the projection operator that projects onto the tangent subspaces orthogonal to  $\eta^a$ . Before continuing with deriving perturbation equations, it will be useful to write down some formulas originating directly from the Killing vector equation  $\nabla_{(a} \eta_{b)} = 0$  in the case of  $\eta^a$  being non-null (i.e.  $F \neq 0$ ). To begin with, we note that the two-form  $\nabla_a \eta_b$  can be decomposed as

$$\nabla_a \eta_b = \eta_{[a} v_{b]} + v_{ab} \quad (3)$$

where

$$\eta^a v_a = 0 \quad \eta^a v_{ab} = 0 \quad v_{(ab)} = 0. \quad (4)$$

Contracting eq. (3) by  $\eta^b$  it follows that

$$v_a = -\nabla_a \ln |F|. \quad (5)$$

In terms of  $\mu_a = F^{-1} \eta_a$ , eq. (3) can be equivalently expressed as

$$\nabla_a \mu_b = \mu_{(a} v_{b)} + F^{-1} v_{ab} \quad (6)$$

which immediately implies that  $v_{ab}$  is proportional to the exterior derivative of  $\mu_a$ ;

$$v_{ab} = \frac{1}{2} F \Omega_{ab} \quad \Omega_{ab} = 2 \nabla_{[a} \mu_{b]}. \quad (7)$$

Hence, we conclude that  $\nabla_a \eta_b$  is completely determined by  $F$ ,  $\nabla_a F$  and the exact two-form  $\Omega_{ab}$  as

$$\nabla_a \eta_b = -\eta_{[a} \nabla_{b]} \ln |F| + \frac{1}{2} F \Omega_{ab}. \quad (8)$$

It should be noted that  $F$  and  $\mu_a$ , and hence their exterior derivatives  $\nabla_a F$  and  $\Omega_{ab}$ , are Lie dragged by  $\eta^a$ . Indeed,

$$\mathcal{L}_\eta F = \eta^a \nabla_a (\eta^b \eta_b) = 2 \eta^a \eta^b \nabla_{(a} \eta_{b)} = 0 \quad \mathcal{L}_\eta \mu_a = F^{-1} \mathcal{L}_\eta \eta_a = 0 \quad (9)$$

where  $\mathcal{L}_\eta \eta_a = 0$  of course follows from  $\mathcal{L}_\eta \eta^a = 0$  and  $\mathcal{L}_\eta g^{ab} = 0$ . Also, it clearly follows from eq. (1) that the projection operator  $\perp_a{}^b$  (with any index positioning) is also annihilated by  $\mathcal{L}_\eta$ . Now, it is easily realized that the Killing vector  $\eta^a$  is hypersurface orthogonal (aligned with a gradient) precisely when the two-form  $\Omega_{ab}$  vanishes. Indeed, the failure of hypersurface orthogonality can in four dimensions be described by the twist vector [8] (sometimes defined with opposite sign as in [9])

$$\Omega^a = -\epsilon^{abcd} \eta_b \nabla_c \eta_d \quad (10)$$

which according to eq. (8) is dually related to  $\Omega_{ab}$  as

$$\Omega^a = -\frac{1}{2} F \epsilon^{abcd} \eta_b \Omega_{cd} \quad \Omega_{ab} = F^{-2} \epsilon_{abcd} \Omega^c \eta^d. \quad (11)$$

Consequently,  $\Omega^a$  also satisfies  $\mathcal{L}_\eta \Omega^a = 0$ .

Let us now consider a one-parameter family of axisymmetric solutions to the Einstein equations, with the aim of linearizing these equations around a particular solution for which  $\Omega_{ab} = 0$ . Referring to  $\lambda$  as the parameter of the family, the notation  $\delta = d/d\lambda$  will be used for derivation of tensor fields. The first order metric perturbation tensor  $\gamma_{ab} = \delta g_{ab}$  turns out to split naturally into two parts according to

$$\begin{aligned} \gamma_{ab} &= {}^+ \gamma_{ab} + {}^- \gamma_{ab} \\ {}^+ \gamma_{ab} &= (\perp_a{}^c \perp_b{}^d + \mu_a \mu_b \eta^c \eta^d) \gamma_{cd} = \delta \perp_{ab} + (\delta F) \mu_a \mu_b \\ {}^- \gamma_{ab} &= 2 \eta^c \mu_{(a} \perp_{b)}{}^d \gamma_{cd} = 2 F \mu_{(a} \delta \mu_{b)} = 2 \eta_{(a} \delta \mu_{b)}. \end{aligned} \quad (12)$$

As always, in the absence of a natural identification of spacetime points along the family of spacetimes (i.e. a preferred one-parameter family of diffeomorphisms),  $\delta$  is only defined up to a gauge transformation

$$\delta \rightarrow \delta + \mathcal{L}_\zeta \quad (13)$$

given in terms of an arbitrary vector field  $\zeta^a$ . However, in the present case it is natural to only consider the families of diffeomorphisms whose associated tangent space mappings takes the Killing vector  $\eta^a$  into “itself”, corresponding to the condition

$$\delta\eta^a = 0 \quad (14)$$

which clearly implies that we only allow for gauge transformations with vector fields restricted to satisfy

$$\mathcal{L}_\zeta\eta^a = -\mathcal{L}_\eta\zeta^a = 0. \quad (15)$$

It then follows from  $\eta^a\mu_a = 1$ ,  $\eta^a\perp_{ab} = 0$  and  $g^{ac}g_{cb} = \delta^a_b$  that

$$\eta^a\delta\mu_a = 0 \quad (16)$$

$$\eta^a\delta\perp_{ab} = 0 \quad (17)$$

$$g_{ac}g_{bd}\delta\perp^{cd} = -\delta\perp_{ab} - 2\eta_{(a}\delta\mu_{b)} \quad (18)$$

which will simplify the calculations to come considerably. To derive what we shall refer to as the axial perturbation equations, we split the Einstein equations in the form  $Z_{ab} := R_{ab} - \kappa(T_{ab} - \frac{1}{2}T^c{}_c g_{ab}) = 0$  into its “components” orthogonal to  $\eta^a$  according to

$$\perp^{ac}\perp^{bd}Z_{cd} = 0 \quad (19)$$

$$\eta^a\eta^b Z_{ab} = 0 \quad (20)$$

$$\perp^{ab}\eta^c Z_{bc} = 0. \quad (21)$$

As we are now about to apply  $\delta$  to these equations, we shall need the general form of  $\delta R_{ab}$ , which reads [9]

$$\delta R_{ab} = -\frac{1}{2}\nabla_a\nabla_b\gamma^c{}_c - \frac{1}{2}\square\gamma_{ab} + \nabla^c\nabla_{(a}\gamma_{b)c} \quad (22)$$

where  $\square = \nabla^a\nabla_a$ . Now, it does not take a too frustrating amount of algebra to realize that when evaluating eqs. (19) - (21) at  $\Omega_{ab} = 0$ , using the above derived formulae, the  ${}^-\gamma_{ab}$  part of the metric perturbation tensor  $\gamma_{ab}$  satisfies eqs. (19) and (20) identically, while on the other hand  ${}^+\gamma_{ab}$  satisfies eq. (21) identically. Hence it follows that the perturbation equations separate  ${}^+\gamma_{ab}$  and  ${}^-\gamma_{ab}$ , implying that those parts of the metric perturbation can be treated separately. We shall here only focus on the  ${}^-\gamma_{ab}$  part, later to be identified with the axial (or odd parity) part when we specialize to the case of a spherically symmetric background. We hence insert  $\gamma_{ab} = {}^-\gamma_{ab}$  into eq. (21), evaluate at  $\Omega_{ab} = 0$  to find the Maxwell-like equation

$$\nabla_b(FQ^{ab}) = \kappa J^a \quad (23)$$

where

$$Q_{ab} = \delta\Omega_{ab} \quad (24)$$

$$J^a = 2\delta T^a \quad T^a = \perp^{ab}\eta^c T_{bc}. \quad (25)$$

It should be noted that since  $\Omega_{ab}$  and  $T^a$  vanish on the background - the former field by assumption, the latter by the twist-free Killing vector  $\eta^a$  being an eigenvector of the Ricci tensor and hence also of the stress-energy tensor (cf. [9]) - it directly follows from eq. (13) that the perturbation fields  $Q_{ab}$  and  $J^a$  are invariant under the gauge transformation (13). This can be recognised as a lemma by Stewart and Walker [10]: the linear perturbation of a quantity is (identification) gauge invariant if that quantity vanishes when unperturbed. Moreover, using  $\delta\eta^a = 0$  it can easily be shown that  $Q_{ab}$  and  $J^a$  share the properties of

their respective nonlinearized fields of being orthogonal to, as well as Lie dragged by,  $\eta^a$ . The perturbation equation (23) should be accompanied by these conditions, as well as the condition of  $Q_{ab}$  being closed, which follows from the closedness of  $\Omega_{ab}$  and the fact that  $\delta$  commutes with exterior differentiation. Moreover, as an integrability condition of eq. (23), it follows that the linearized matter current  $J^a$  is conserved,

$$\nabla_a J^a = 0. \quad (26)$$

In the next section we will in fact not be working with the Maxwell-like equation (23) directly, but rather with its dual equation

$$2\nabla_{[a}Q_{b]} = -\kappa J_{ab} \quad (27)$$

where

$$Q^a = \delta\Omega^a \quad J_{ab} = \epsilon_{abcd}J^c\eta^d. \quad (28)$$

The linearized twist vector  $Q^a$  as well as the dual matter field  $J_{ab}$  can easily be shown to also have the properties of being orthogonal to and Lie dragged by  $\eta^a$ . It should be noted that the duality relations (11) for  $\Omega_{ab}$  and  $\Omega^a$  holds also for their linearized versions  $Q_{ab}$  and  $Q^a$ , which is due to the fact that the perturbation equations are evaluated on a background where  $\Omega_{ab}$  and  $\Omega^a$  vanishes. Moreover, the above relation between  $J_{ab}$  and  $J^a$  can be inverted into

$$J^a = -\frac{1}{2}F^{-1}\epsilon^{abcd}\eta_b J_{cd}. \quad (29)$$

Finally, in this dual formulation, the additional equations  $\nabla_{[a}Q_{bc]} = 0$  and  $\nabla_a J^a = 0$  are replaced by

$$\nabla_a(F^{-2}Q^a) = 0 \quad (30)$$

$$\nabla_{[a}J_{bc]} = 0. \quad (31)$$

### 3 Axial gravitational perturbations

We shall now restrict the reduced perturbation equations derived in the previous section to the case of a spherically symmetric background perturbed into an axisymmetric neighbour. As mentioned in the introduction, the Killing vector  $\eta^a$  will thus be chosen as the generator of the axisymmetry, reducing simply to an SO(3) generator for the background. The metric of the background geometry will be written in the standard form

$$ds^2 = d\sigma^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (32)$$

where  $d\sigma^2$  is the induced Lorentzian two-metric on the space orthogonal to the SO(3) group orbits and  $r$  is the Schwarzschild radius. For simplicity, we will arrange the coordinates so that the Killing vector  $\eta^a$  will coincide with  $(\partial/\partial\phi)^a$ . Note that the scalar  $F = \eta^a\eta_a$  on this background takes the form

$$F = (r \sin\theta)^2. \quad (33)$$

To reduce the perturbation equations to two-dimensional ones by separating out the dependence of the remaining angular variable  $\theta$ , we start off by noting that no restriction is implied by assuming that the two-form  $J_{ab}$ , which by eq. (31) is closed, can be given in terms of a potential  $Y_a$  which is orthogonal to the SO(3) orbits (i.e. whose  $\theta$  component is vanishing). Thus we set

$$J_{ab} = 2\nabla_{[a}Y_{b]}. \quad (34)$$

It then follows from eq. (27) that the linearized twist vector  $Q^a$  can be written in terms of  $Y_a$  and a scalar  $\Phi$  as

$$Q_a = \nabla_a\Phi - \kappa Y_a. \quad (35)$$

Turning to eq. (30) which corresponds to the closedness condition for  $Q_{ab}$ , the  $\theta$  dependence can be separated out by setting

$$\Phi = C(\theta)r\psi \quad Y_a = C(\theta)X_a \quad (36)$$

where the scalar  $\psi$  and the one-form  $X_a$  have no angular dependence, i.e. they are Lie dragged by all  $\text{SO}(3)$  generators. Inserting eq. (35) into eq. (30) leads to a separation into an ordinary differential equation for  $C(\theta)$  as well a two-dimensional wave equation for  $\psi$  coupled to  $X_a$ . The equation for  $C(\theta)$  will be discussed further in the appendix. Here we merely note that it is solved by setting

$$C(\theta) = G_{l+2}^{-3/2}(\cos \theta) \quad (37)$$

with  $G_{l+2}^{-3/2}(y)$  being the ultraspherical (or Gegenbauer) polynomial. The wave equation can be covariantly written in form

$$(\mathcal{D}_a \mathcal{D}^a - U)\psi = \kappa S \quad (38)$$

where  $\mathcal{D}_a$  is the connection of the Lorentzian two-metric  $d\sigma^2$ . The potential  $U$  can be put in the formally simple form

$$U = \frac{1}{2}\kappa\tau - \frac{6m}{r^3} + \frac{l(l+1)}{r^2} \quad (39)$$

with  $\tau$  invariantly defined (given the spherical symmetry) by  $\frac{1}{2}\kappa\tau$  being the eigenvalue function of the Ricci tensor with respect to the eigenvector  $\eta^a$  (or any other  $\text{SO}(3)$  generator), i.e.

$$R^a{}_b \eta^b = \frac{1}{2}\kappa\tau \eta^a. \quad (40)$$

The reason for defining  $\tau$  with the factor  $\frac{1}{2}\kappa$  inserted as above is that the background Einstein equations imply that  $\tau$  is simply minus the trace of the  $2 \times 2$  block of the stress-energy tensor orthogonal to the  $\text{SO}(3)$  orbits, an interpretation which has a clearer physical significance. The function  $m$  is the spherically symmetric mass function [11], invariantly defined in terms of the Schwarzschild radius  $r$  according to

$$\nabla_a r \nabla^a r = 1 - \frac{2m}{r}. \quad (41)$$

The source term  $S$  on the left hand side of eq. (38) is given by

$$S = r\mathcal{D}_a(r^{-2}X^a) \quad (42)$$

and is thus directly related to the perturbation of the stress-energy tensor. We will have more to say about the wave equation (38), but this will be postponed to section 4.

Having found the general solution to the conserved current condition  $\nabla_a J^a = 0$  by introducing a gauge potential  $Y_a$  for the dual field  $J_{ab}$  according to eq. (34), it is of interest to transform this solution back to  $J^a$  which is more directly related to the matter perturbations. Using eqs. (29), (34) and (36), it follows that  $J^a$  can be expressed as

$$J^a = (r^2 \sin \theta)^{-1} [C'(\theta)\mathcal{J}^a - C(\theta)\mathcal{J}(\partial/\partial\theta)^a] \quad (43)$$

where

$$\mathcal{J}^a = \epsilon^{ab}X_b \quad \Leftrightarrow \quad X_a = \epsilon_{ab}\mathcal{J}^b \quad \mathcal{J} = \epsilon^{ab}\mathcal{D}_a X_b \quad (44)$$

with  $\epsilon_{ab}$  being the Levi-Civita volume form of the induced two-metric. It directly follows that  $\nabla_a J^a = 0$  can be solved by putting  $J^a$  in the form given by eq. (43) and requiring that  $\mathcal{J}^a$  and  $\mathcal{J}$  satisfy

$$\mathcal{D}_a \mathcal{J}^a = \mathcal{J} \quad (45)$$

as implied by eqs. (44). It should be stressed that since the form  $J^a$  may be restricted by the types of matter studied, it may not be practical to solve the conserved current condition by expressing  $\mathcal{J}^a$  and  $\mathcal{J}$  in terms of some  $X_a$  according to eqs. (44). Rather, the two-dimensional equation (45) in general has to be considered as nontrivial equations of motion for the perturbed matter fields. Hence it is appropriate to note that the source term of the wave equation (38) can be expressed in terms of  $\mathcal{J}^a$  rather than  $X_a$  as

$$S = r\epsilon^{ab}\mathcal{D}_a(r^{-2}\mathcal{J}_b). \quad (46)$$

Let us now discuss Israel's junction conditions [12] of the derived perturbation equations. Returning first to the general, not necessarily spherically symmetric case, we assume that we are given a non-null hypersurface  $\Sigma$  with unit normal  $n^a$ ,  $n^a n_a = \epsilon = \pm 1$ , which may be given an arbitrary smooth continuation off of  $\Sigma$ . For practical purposes we need only consider the case when  $\eta^a$  is tangent to the hypersurface, i.e.  $\eta^a n_a = 0$ . The fields to be matched across  $\Sigma$  are the first and second fundamental forms defined as

$$j_{ab} = g_{ab} - \epsilon n_a n_b \quad (47)$$

$$k_{ab} = j_a^c j_b^d \nabla_c n_d. \quad (48)$$

By using a gauge transformation generated by a vector field  $\zeta^a$  satisfying

$$\eta^a \zeta_a = 0 \quad \mathcal{L}_\eta \zeta^a = 0 \quad (49)$$

we may set  $\delta n_a$  to zero. Such a gauge transformation only affects the polar part of the perturbations, since it leaves the axial field  $\delta\mu_a = \delta(F^{-1}\eta_a)$  invariant. Hence we are able to make the gauge choice  $\delta n_a = 0$  without making any gauge fixing in the axial sector that we are considering. For simplicity, we therefore set  $\delta n_a = 0$  to find that

$$\delta j_{ab} = \delta g_{ab} = \gamma_{ab}. \quad (50)$$

The relevant projection for the axial perturbation junction condition is

$$\perp_a^b \eta^c \delta j_{bc} = F \delta\mu_a \quad (51)$$

from which we conclude that all components of  $\delta\mu_a$  should be matched. Note, however, that  $\delta\mu_a$  is not a gauge invariant field, since  $\mu_a$  is not zero on the unperturbed spacetime. Namely, under a gauge transformation generated by a vector field  $\zeta^a$  such that

$$\zeta^a = f\eta^a \quad \eta^a \nabla_a f = 0 \quad (52)$$

one finds that  $\delta\mu_a$  transforms as

$$\delta\mu_a \rightarrow \delta\mu_a + \mathcal{L}_\zeta \mu_a = \delta\mu_a + f\mathcal{L}_\eta \mu_a + (\eta^b \mu_b) \nabla_a f = \delta\mu_a + \nabla_a f. \quad (53)$$

As far as the second fundamental form is concerned, the relevant quantity to consider and match across  $\Sigma$  is

$$\delta(\perp_a^b \eta^c k_{bc}) = -\frac{1}{2}FQ_{ab}n^b = -\frac{1}{2}F^{-1}\epsilon_{abcd}n^b Q^c \eta^d. \quad (54)$$

Let us now specify these matching conditions to the spherically symmetric case. Excluding the nonradiative case  $l = 1$  which will not be treated here, it is implied by eqs. (35) - (42) that  $\delta\mu_a$  and  $Q_{ab}$  take the forms

$$\delta\mu_a = -\frac{C'(\theta)}{(l+2)(l-1)r^2 \sin^3 \theta} \epsilon_a^b [\mathcal{D}_b(r\psi) - \kappa X_b] + \nabla_a f \quad (55)$$

$$Q_{ab} = \frac{\psi C'(\theta)}{r^3 \sin^3 \theta} \epsilon_{ab} + \frac{2C(\theta)}{r^2 \sin^3 \theta} (\nabla_{[a} \theta) \epsilon_{b]}^c [\mathcal{D}_c(r\psi) - \kappa X_c] \quad (56)$$

where the term  $\nabla_a f$  in the expression for  $\delta\mu_a$  corresponds to the freedom to make the gauge transformation (53). Now, since  $\delta\mu_a$  is to be continuous and since  $\nabla_a f = \mathcal{D}_a f + \partial f / \partial \theta \nabla_a \theta$ , it directly follows that  $\partial f / \partial \theta$

has to continuous. Furthermore, the possibility of adding to  $f$  a  $\theta$ -independent term with discontinuous gradient is excluded by the fact that  $C'(\theta)/\sin^3\theta$  is not constant for  $l \neq 1$ . Consequently, the matching of  $\delta\mu_a$  leads to the condition that  $\epsilon_a{}^b[\mathcal{D}_b(r\psi) - \kappa X_b]$  and hence  $\mathcal{D}_a(r\psi) - \kappa X_a$  be matched. For all practical purposes we can assume that  $n^a$  is orthogonal to the SO(3) orbits, implying that  $n^a\nabla_a\theta = 0$  and hence

$$Q_{ab}n^b = \frac{\psi C'(\theta)}{r^3 \sin^3\theta} \epsilon_{ab}n^b + \frac{C(\theta)}{r^2 \sin^3\theta} n^b \epsilon_b{}^c [\mathcal{D}_c(r\psi) - \kappa X_c] \nabla_a\theta. \quad (57)$$

It directly follows that the matching of  $Q_{ab}n^b$  leads to the further condition that the two-scalar  $\psi$  itself should be matched.

## 4 Klein-Gordon and Maxwell fields

In this section, we shall consider Klein-Gordon and Maxwell test fields on a general spherically symmetric background and, as was done for the axial gravitational perturbations treated in the previous section, reduce the equations of motion to two-dimensional wave equations.

Starting with the Klein-Gordon case, the equations of motion is taken to be

$$\square\Phi = \varepsilon_0 j \quad (58)$$

where an unspecified source scalar  $j$  multiplied by a coupling constant  $\varepsilon_0$  has been added to the free massless Klein-Gordon equation. Restricting to axisymmetric fields,  $\Phi$  as well as  $j$  will be assumed to be Lie dragged by  $\eta^a$ , one of the SO(3) generators. Referring to the form (32) for the general spherically symmetric metric, the  $\theta$  dependence can straightforwardly be separated out according to

$$\Phi = C(\theta)r^{-1}\psi \quad (59)$$

$$j = C(\theta)r^{-1}S \quad (60)$$

with  $\psi$  and  $S$  being SO(3) invariant scalars. Taking  $C(\theta)$  to be the Legendre polynomial  $P_l(\cos\theta)$ , the Klein-Gordon equation reduces to the covariant two-dimensional equation

$$(\mathcal{D}_a\mathcal{D}^a - U)\psi = \varepsilon_0 S \quad (61)$$

which is formally identical to the eq. (38), but with the first two terms in the potential  $U$  having different coefficients;

$$U = -\frac{1}{2}\kappa\tau + \frac{2m}{r^3} + \frac{l(l+1)}{r^2}. \quad (62)$$

As for Maxwells equations

$$\nabla_b F^{ab} = \varepsilon_1 j^a \quad (63)$$

$$\nabla_{[a} F_{bc]} = 0 \quad (64)$$

$$\nabla_a j^a = 0 \quad (65)$$

with the electromagnetic field strength  $F_{ab}$  and the current  $j^a$  assumed to be Lie dragged by  $\eta^a$ , the hypersurface orthogonality of  $\eta^a$  makes it natural to make the splitting

$$F_{ab} = P_{ab} + 2\mu_{[a}S_{b]} \quad (66)$$

$$j_a = J_a + J\mu_a \quad (67)$$

where all introduced fields are assumed to be orthogonal to  $\eta^a$  as well as annihilated by  $\mathcal{L}_\eta$ . Indeed, using  $\nabla_{[a}\mu_{b]} = 0$  leads to a decoupling into the two sets of equations

$$F\nabla_a(F^{-1}S^a) = \varepsilon_1 J \quad (68)$$

$$\nabla_{[a}S_{b]} = 0 \quad (69)$$



and

$$\nabla_b P^{ab} = \varepsilon_1 J^a \quad (70)$$

$$\nabla_{[a} P_{bc]} = 0 \quad (71)$$

$$\nabla_a J^a = 0 \quad (72)$$

corresponding to the axial and polar sector, respectively.

Starting with the simpler axial case, we solve eq. (69) by setting

$$S_a = \nabla_a \Phi. \quad (73)$$

The angular dependence can now be immediately be separated out, this time according to

$$\Phi = C(\theta)\psi \quad (74)$$

$$J = C(\theta)S \quad (75)$$

with  $C(\theta)$  taken to be the ultraspherical polynomial  $G_{l+1}^{-1/2}(\cos \theta)$  and  $\psi$  and  $S$  having no angular dependence. The resulting two-dimensional equation again takes the form (38)

$$(\mathcal{D}_a \mathcal{D}^a - U)\psi = \varepsilon_1 S \quad (76)$$

but with the potential  $U$  in this case only consisting of the centrifugal term

$$U = \frac{l(l+1)}{r^2}. \quad (77)$$

Turning to the polar case, which clearly can be treated similarly to the axial gravitational perturbations, we introduce the fields  $P^a$  and  $J_{ab}$  dual to  $P_{ab}$  and  $J^a$  according to

$$P^a = -\frac{1}{2}\epsilon^{abcd}\eta_b P_{cd} \quad P_{ab} = F^{-1}\epsilon_{abcd}P^c\eta^d \quad (78)$$

$$J_{ab} = \epsilon_{abcd}J^c\eta^d \quad J^a = -\frac{1}{2}F^{-1}\epsilon^{abcd}\eta_b J_{cd}. \quad (79)$$

This turns eqs. (70) - (72) into

$$2\nabla_{[a} P_{b]} = -\varepsilon_1 J_{ab} \quad (80)$$

$$\nabla_a (F^{-1}P^a) = 0 \quad (81)$$

$$\nabla_{[a} J_{bc]} = 0. \quad (82)$$

We start off by solving the closedness condition for  $J_{ab}$  according to

$$J_{ab} = 2\nabla_{[a} Y_{b]} \quad (83)$$

with the condition that the gauge potential  $Y_a$  be orthogonal to the  $\text{SO}(3)$  orbits. According to eq. (80), the field  $P_a$  can be written as

$$P_a = \nabla_a \Phi - \varepsilon_1 Y_a \quad (84)$$

for some scalar potential  $\Phi$ . To reduce the remaining equation (81) into a two-dimensional one, we set

$$\Phi = C(\theta)\psi \quad Y_a = C(\theta)X_a \quad (85)$$

with  $C(\theta) = G_{l+1}^{-1/2}(\cos \theta)$  and with  $\psi$  and  $X_a$  being  $\text{SO}(3)$  invariant fields. Again, we obtain a two-dimensional wave equation of the form (38). As for the axial Maxwell case, the potential  $U$  is given by eq. (77), while the source function  $S$  has the form

$$S = \mathcal{D}_a X^a = \epsilon^{ab}\mathcal{D}_a \mathcal{J}_b. \quad (86)$$

The field  $\mathcal{J}_a$  here refers to the form of  $J^a$ ,

$$J^a = (r^2 \sin \theta)^{-1} [C'(\theta) \mathcal{J}^a - C(\theta) \mathcal{J} (\partial/\partial \theta)^a] \quad (87)$$

where, as implied by the duality between  $J^a$  and  $J_{ab}$ ,

$$\mathcal{J}^a = \epsilon^{ab} X_b \quad \mathcal{J} = \epsilon^{ab} \mathcal{D}_a X_b. \quad (88)$$

In analogy with the axial gravitational perturbations, the conserved current condition  $\nabla_a J^a$  is thus found to reduce to the two-dimensional equation

$$\mathcal{D}_a \mathcal{J}^a = \mathcal{J}. \quad (89)$$

Having found that the linear equations of motion for Klein-Gordon fields, Maxwell fields (axial as well as polar) and axial gravitational perturbations are all governed by a two-dimensional wave equation of the form

$$(\mathcal{D}^a \mathcal{D}_a - U) \psi = \varepsilon S \quad (90)$$

it is interesting to note that the potential  $U$  can be naturally split into a spin dependent part  $U_s$  and an angular momentum dependent part  $U_l$  as

$$U = U_s + U_l \quad (91)$$

$$U_s = (s-1) \left[ \frac{1}{2} \kappa \tau - (s+1) \frac{2m}{r^3} \right] \quad (92)$$

$$U_l = \frac{l(l+1)}{r^2} \quad (93)$$

with  $s$  taking the values 0, 1 and 2 for scalar, electromagnetic and linearized gravity fields, respectively. Note also that the value of the coupling constant  $\varepsilon$  on the left hand side of eq. (90) depends on the spin  $s$  as well;  $\varepsilon = \varepsilon_s$  with  $\varepsilon_2 = \kappa$ . In geometrical units  $2\varepsilon_1 = \varepsilon_2 = 8\pi$ . Now, to connect more closely to previous works on various static backgrounds, let us choose as coordinates two harmonically conjugate functions [9]  $t$  and  $x$  to make the two-metric  $d\sigma^2$  take the manifestly conformally flat form

$$d\sigma^2 = e^{2\nu} (-dt^2 + dx^2). \quad (94)$$

This turns the wave equation (90) into the form

$$\left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V \right) \psi = \varepsilon H \quad (95)$$

where the potential  $V$  and the source  $H$  are simply given by rescaling the invariant potential  $U$  and source  $S$  with the conformal factor  $e^{2\nu}$ , i.e.

$$V = e^{2\nu} U \quad H = e^{2\nu} S \quad (96)$$

implying of course that  $V$  also naturally splits according to  $V = V_s + V_l$ . The coordinates  $t$  and  $x$  are clearly not uniquely defined, since the freedom of choosing them is in correspondance with the infinite dimensional conformal group in two dimensions, but in the static case there is a natural particular choice. Starting out from Schwarzschild coordinates

$$d\sigma^2 = -e^{2\nu} dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 \quad (97)$$

with  $\nu$  and  $m$  being functions of  $r$  only, the harmonically conjugate variables can preferably be taken to be the static time  $t$  and the Regge-Wheeler (or tortoise) radial variable

$$r_* = \int e^{-\nu} (1 - 2m/r)^{-1/2} dr \quad (98)$$

which gives  $d\sigma^2 = e^{2\nu}(-dt^2 + dr_*^2)$ . In this case, we may turn eq. (95) into a one-dimensional Schrödinger equation (in general with a source term) by separating out the time dependence in the standard manner according to

$$\psi = e^{i\omega t} Z(r_*) \quad H = e^{i\omega t} Y(r_*) \quad (99)$$

which results in

$$\left(-\frac{d^2}{dr_*^2} + V - \omega^2\right) Z = -\varepsilon Y. \quad (100)$$

Clearly, besides the source  $Y$ , this is a standard one-dimensional Schrödinger equation with the squared frequency  $\omega^2$  corresponding to the energy.

The splitting of the potential  $V$  into a spin dependent and an angular momentum dependent part has previously been noted in the case of a Schwarzschild background (cf. [7]), for which the matter term  $\tau$  of course vanishes and the mass function  $m$  is a constant, but it does not appear to have been known to generalize to the case of a general spherically symmetric background in the simple manner found here. Indeed, whether or not the background is static, the only formal modification of the potential  $V$  compared to the axial gravitational wave potential found by Chandrasekhar and Ferrari for static perfect fluid backgrounds [5], is the simplest possible: the matter combination  $\rho - p$  has to be replaced by  $\tau$ , minus the trace of the  $2 \times 2$  block of the stress-energy tensor that is  $SO(3)$  orbit orthogonal. Clearly,  $\tau$  can be expressed as  $\rho - p_\perp$ , with  $\rho$  and  $p_\perp$  being the energy density and radial pressure measured by an arbitrary radially moving observer. It should be noted, however, that whereas the combination  $\rho - p_\perp$  is invariant under a change of such an observer, the same does in general not hold true for the quantities  $\rho$  and  $p_\perp$  themselves.

## 5 An application to fluids with viscosity

As a concrete and physically relevant example, we shall here consider the general Regge-Wheeler type wave equation for axial gravitational perturbations of spherically symmetric backgrounds, in the case when the matter can be described as a fluid with viscosity in the standard relativistic manner. The stress energy tensor for a viscous fluid is [13]

$$T_{ab} = \rho u_a u_b + (p - \zeta \Theta) h_{ab} - 2\eta \sigma_{ab} \quad (101)$$

where the various fields involved are the fluid four-velocity  $u^a$ , the energy density  $\rho$ , the pressure  $p$ , the coefficient of bulk viscosity  $\zeta$ , the coefficient of dynamic (or shear) viscosity  $\eta$ , the projection operator  $h_{ab} = u_a u_b + g_{ab}$ , the expansion  $\Theta = \nabla_a u^a$  and the shear tensor  $\sigma_{ab} = (\nabla_c u_{(a} h_{b)})^c - \frac{1}{3} \Theta h_{ab}$ . For the background solution, the four-velocity  $u^a$  will be assumed to be orthogonal to  $\eta^a$ , meaning that the matter is not rotating around the symmetry axis. In particular, in the spherically symmetric case the fluid will be assumed to be radially moving, thus automatically satisfying  $\eta^a u_a = 0$ . We now note that the vector  $T^a = \perp^{ab} \eta^c T_{bc}$  takes the form

$$T^a = (\rho + p - \zeta \Theta) \eta^c u_c \perp^{ab} u_b - 2\eta \perp^{ab} \eta^c \sigma_{bc}. \quad (102)$$

Applying the perturbation operator  $\delta$  to this vector, we arrive at the following form of the linearized matter current  $J^a$ :

$$\frac{1}{2} J^a = \delta T^a = (\rho + p - \zeta \Theta - 2\eta \sigma) \beta u^a + \eta [F Q^{ab} u_b - F h^{ab} \nabla_b (F^{-1} \beta) - \beta \dot{u}^a] \quad (103)$$

where

$$\beta = \eta^a \delta u_a \quad (104)$$

$$\sigma = F^{-1} \eta^a \eta^b \sigma_{ab}. \quad (105)$$

The popping up of the two-form  $Q_{ab}$  in the expression for  $J^a$  originates in the perturbation of the shear  $\sigma_{ab}$ , being defined in terms of the covariant derivative of the four-velocity  $u_a$ . More precisely, we have used

$$\delta(\nabla_a u_b) = \nabla_a \delta u_b + u_c \delta \Gamma^c_{ab} \quad \delta(\perp^a_c \perp^d_b \Gamma^c_{de} \eta^e) = -\frac{1}{2} F Q^a_b \quad (106)$$

where  $\Gamma^c_{ab}$  refers to the Christoffel symbols. As we now specialise to the case of a spherically symmetric background, the decomposition of  $J^a$  in accordance with eq. (43) is achieved by decomposing the scalar  $\eta^a \delta u_a$  according to

$$\eta^a \delta u_a = \sin^{-1} \theta C'(\theta) \beta \quad (107)$$

where  $C(\theta)$ , as in section 3, refers to the Gegenbauer polynomial  $G_{l+2}^{-3/2}(\cos \theta)$ , while  $\beta$  is a scalar without angular dependence. The resulting expressions for the two-dimensional fields  $\mathcal{J}^a$  and  $\mathcal{J}$  are

$$\mathcal{J}^a = 2r^2(\rho + p - \zeta\Theta - 2\eta\sigma) \beta u^a - 2\eta r [\psi + r^3 n^b \mathcal{D}_b(r^{-2}\beta) + r \dot{u} \beta] n^a \quad (108)$$

$$\mathcal{J} = -2\eta \{n^a \mathcal{D}_a(r\psi) + [(l+2)(l-1) + 2\kappa r^2(\rho + p - \zeta\Theta - 2\eta\sigma)] \beta\} \quad (109)$$

where  $n_a = u^b \epsilon_{ba}$ , implying that  $n^a$  is the (up to sign) unique unit space-like vector which is orthogonal to both  $u^a$  as well as the SO(3) orbits. We have also introduced the notation  $\dot{u}$  for the norm of the acceleration  $\dot{u}^a$ , and used that  $\dot{u}^a = \dot{u} n^a$ . For simplicity, we shall now further restrict the background by requiring that it be static. Using Regge-Wheeler variables, the two-metric  $d\sigma^2$  and the unit vectors  $u^a$  and  $n^a$  take on the forms

$$d\sigma^2 = e^{2\nu}(-dt^2 + dr_*^2) \quad (110)$$

$$u^a = e^{-\nu}(\partial/\partial t)^a \quad (111)$$

$$n^a = e^{-\nu}(\partial/\partial r_*)^a \quad (112)$$

with the gravitational potential  $\nu$  being a function of  $r_*$  only. The time dependence of the perturbation fields is straightforwardly separated out according to

$$\psi = e^{i\omega t} Z(r_*) \quad \beta = e^{i\omega t} B(r_*). \quad (113)$$

The assumption of staticity implies that both the expansion  $\Theta$  as well as the shear  $\sigma_{ab}$  vanishes on the unperturbed spacetime. This means that both  $\Theta$  and  $\sigma$  will be set to zero in eqs. (108) and (109), which inserted into the matter current integrability condition (45) leads to the ordinary differential equation

$$\left(-\frac{d^2}{dr_*^2} - e^\nu \dot{u} \frac{d}{dr_*} + W\right) B = e^{2\nu} \frac{\dot{u}}{r} Z \quad (114)$$

where, after using the unperturbed Einstein equations,

$$W = e^{2\nu} \left[ \frac{\kappa}{2}(\rho + 5p) + \frac{2m}{r^3} + \frac{l(l+1)}{r^2} - 2\sqrt{1 - \frac{2m}{r}} \frac{\dot{u}}{r} \right] + \frac{i\omega}{\eta} e^\nu (\rho + p). \quad (115)$$

In obtaining eq. (114), we have multiplied eq. (45) by the coefficient of shear viscosity  $\eta$ . The case  $\eta = 0$  is hence recovered from eq. (114) by first multiplying by  $\eta$  and thereafter setting  $\eta = 0$ . This results in the simple condition

$$\omega B = 0 \quad (116)$$

meaning that for vanishing coefficient of shear viscosity, the perturbation is either stationary ( $\omega = 0$ ) or can be interpreted as a pure gravitational wave not interacting with the matter ( $B = 0$ ). This is in agreement with Chandrasekhar and Ferrari's work on axial perturbations of isentropic perfect fluids [5], but note that

we here have not assumed that the coefficient of bulk viscosity vanishes, so the fluid need not be an isentropic one for the same result to hold.

Returning to the case  $\eta \neq 0$ , the Schrödinger equation (100) (with  $s = 2$ ), the source function  $Y$  can be written as

$$Y = -2r \frac{d}{dr_*} [(\rho + p)e^\nu B] + 2i\omega\eta \left[ r^3 e^{-\nu} \frac{d}{dr_*} (r^{-2} e^\nu B) + e^\nu Z \right] \quad (117)$$

where it has been used that  $\dot{u} = e^{-\nu} d\nu/dr_*$  to put  $Y$  into a compact form. Since  $Y$  contains the wave function  $Z$ , it is clearly appropriate to rewrite eq. (100) by moving the term in question to the left hand side, with the result

$$\left[ -\frac{d^2}{dr_*^2} + V - \omega(\omega - 2i\kappa\eta e^\nu) \right] Z = -\kappa\tilde{Y} \quad (118)$$

where

$$V = e^{2\nu} \left[ \frac{1}{2}\kappa(\rho - p) - \frac{6m}{r^3} + \frac{l(l+1)}{r^2} \right] \quad (119)$$

$$\tilde{Y} = -2r \frac{d}{dr_*} [(\rho + p)e^\nu B] + 2i\omega\eta r^3 e^{-\nu} \frac{d}{dr_*} (r^{-2} e^\nu B). \quad (120)$$

To summarize, the axial perturbations of a static spherically symmetric viscous fluid are governed by the system of two coupled ordinary differential equations (118) and (114). To solve the equations numerically, one starts at the center, using as initial conditions the following expansions in the Schwarzschild radius of the solution with regular center:

$$Z = r^{l+1}(Z_0 + Z_1 r^2 + \dots) \quad (121)$$

$$B = r^{l+1}(B_0 + B_1 r^2 + \dots) \quad (122)$$

where

$$(2l+3)Z_1 = \left\{ \frac{\kappa}{4}(l+2) \left[ \frac{1}{3}(2l-1)\rho_0 - p_0 \right] - \frac{1}{2}\omega_0(\omega_0 - 2i\kappa\eta) \right\} Z_0 \\ - \kappa[(l+1)(\rho_0 + p_0) - (l-1)i\omega_0\eta] B_0 \quad (123)$$

$$(2l+3)B_1 = \left\{ \frac{\kappa}{4} \left[ \left( 1 + \frac{2}{3}l(l+1) \right) \rho_0 - (2l-1)p_0 \right] + \frac{i}{2} \frac{\omega_0}{\eta} (\rho_0 + p_0) \right\} B_0 \\ - \frac{\kappa}{12}(\rho_0 + 3p_0)Z_0 \quad (124)$$

$$\omega_0 = e^{-\nu_0} \omega. \quad (125)$$

To arrive at these expansion coefficients we have used

$$\rho = \rho_0 + \rho_1 r^2 + \dots \\ p = p_0 + p_1 r^2 + \dots \\ m = \frac{1}{6}\kappa\rho_0 + m_1 r^2 + \dots \\ \nu = \nu_0 + \frac{1}{12}\kappa(\rho_0 + 3p_0)r^2 + \dots \quad (126)$$

which follows from the background Einstein equations. The integration of the perturbation equations should be continued up to the stellar surface at  $r = R$ , with  $R$  denoting the Schwarzschild radius of the star. While

the function  $B$  has no vacuum analogue, the integration of the function  $Z$  should be further continued into the vacuum region  $r > R$  where eq. (118) is replaced by the original Regge-Wheeler equation

$$\left[ -\frac{d^2}{dr_*^2} + \left( 1 - \frac{2M}{r} \right) \left( -\frac{6M}{r^3} + \frac{l(l+1)}{r^2} \right) - \omega^2 \right] Z = 0 \quad (127)$$

where  $M = m|_{r=R}$ . The general condition that  $\psi$  and  $\mathcal{D}_a(r\psi) - \kappa X_a$  be matched across  $r = R$  boils down to

$$Z|_{r \rightarrow R^-} = Z|_{r \rightarrow R^+} \quad (128)$$

$$\left[ \frac{d}{dr_*}(rZ) + 2\kappa e^\nu r^2(\rho + p)B \right]_{r \rightarrow R^-} = \left[ \frac{d}{dr_*}(rZ) \right]_{r \rightarrow R^+} \quad (129)$$

$$\left[ \frac{d}{dr_*}(r^{-2}e^\nu B) \right]_{r \rightarrow R^-} = 0. \quad (130)$$

Furthermore, if the condition of purely outgoing gravitational waves at infinity is imposed, corresponding to  $Z \propto e^{-i\omega r_*}$  as  $r \rightarrow \infty$ , one is lead to a discrete gravitational wave spectrum for the system, much in the same way as for perfect fluids stars. The difference is that the gravitational waves (closely related to the function  $Z$ ) in this case interacts with fluid shear waves (closely related to the function  $B$ ) in the stellar interior.

## 6 Concluding remarks

So far, most of the works on gravitational perturbations of spherically symmetric spacetimes have dealt with black holes or static perfect fluid stellar models, with focus on the determination of the quasinormal modes or means to excite them through various astrophysical processes (cf. [7] for a review). Gravitational radiation emitted by collapsing systems has also been studied to some extent, notable examples being the work by Cunningham, Price and Moncrief on Oppenheimer-Snyder collapse [14, 15] and by Seidel and Moore on more general collapsing perfect fluids [16]. Since Choptuik's discovery of critical phenomena in gravitational collapsing systems [17] and the subsequent development of the subject into a brand new branch of general relativity, an increasing number of different types of matter has been studied in the context of, in particular, spherically symmetric gravitational collapse. Since analysing perturbations of the critical solutions is a crucial part of the game, it is clearly desirable to have a general framework for perturbations of spacetimes with spherical symmetry, but without any other restriction on neither the properties of the background nor on the perturbations themselves. It is hoped that this work, being complementary to Gerlach and Sengupta's [6], provides such a framework.

In this paper, we have put the axial gravitational perturbations on an equal footing with Klein-Gordon and Maxwell test fields through the unified two-dimensional wave equation presented in the previous section. However, the more complicated polar gravitational perturbations have not been discussed at all. Although some attempts have been made to reduce the polar equations into a neat form, we have so far not been able to take the equations any further than Gerlach and Sengupta. However, judging from the vacuum case - a Schwarzschild background perturbed into a more general vacuum neighbour - one may suspect that a certain illuminating reformulation of the system of two-dimensional polar equations may be possible in general. Indeed, as discovered by Chandrasekhar [2], the vacuum axial and polar equations are related by what was later to be recognized as a supersymmetry duality [18], more precisely meaning that the Regge-Wheeler and Zerilli potentials for the axial respectively polar modes are supersymmetric partners with the implication that it is only necessary to investigate the properties of the former, simpler one. This indicates that it could be possible that the polar equations for the general spherically symmetric case can be reduced to an equation formally identical to the generalized Regge-Wheeler equation found here for the axial case, combined with a set of equations directly related to the matter perturbations. Whether or not this is merely wishful thinking remains an issue for further investigation.

We have looked into the case of viscous fluid stellar models in some particular detail and derived a system of equations for the axial perturbations that can be directly applied for numerical studies. It would be interesting to explicitly calculate the w-modes for some realistic models to see how the spectrum depends

on the shear viscosity  $\eta$  which enters the equations. It would also be of interest to do a similar analysis for other types of matter sources that are more general than perfect fluids, such as elastic materials as described in a general relativistic setting by Carter and Quintana [19].

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## Appendix: differential equation for the angular function $C(\theta)$

For all integer spin fields studied in this paper, the separation of the dependence on the angular variable  $\theta$  leads to the ordinary differential equation

$$\left( \frac{d}{d\theta} \sin^{1-2s} \theta \frac{d}{d\theta} + k \sin^{1-2s} \theta \right) C(\theta) = 0 \quad (131)$$

where  $s = 0, 1, 2$  is the spin of the field and  $k$  is a separation constant. Setting  $k = (l + s)(l - s + 1)$ , the general solution can be written down as

$$C(\theta) = (1 - y^2)^{s/2} [C_1 P_l^s(\cos \theta) + C_2 Q_l^s(\cos \theta)] \quad (132)$$

where  $P_l^s(y)$  and  $Q_l^s(y)$  are the associated Legendre functions of, respectively, first and second kind. To obtain solutions that are regular at  $y = \pm 1$ , we must set  $C_2$  to zero and  $l$  to an integer (which can be taken to be nonnegative since  $(l + s)(l - s + 1)$  is invariant under  $l \rightarrow -l - 1$ ). The solution then becomes the ultraspherical (or Gegenbauer) polynomial  $G_{l+s}^{1/2-s}(y)$ , which for the radiative values of  $l$ , i.e.  $l \geq s$ , is related to the Legendre polynomial  $P_l(y)$  as

$$G_{l+s}^{1/2-s}(y) = N_l^s (1 - y^2)^s \frac{d^s}{dy^s} P_l(y) \quad (133)$$

where  $N_l^s$  is a normalization factor given by

$$\begin{aligned} N_l^0 &= 1 \\ N_l^s &= \frac{\prod_{j=1}^s (2j - 1)}{\prod_{j=1-s}^s (l + j)} \quad (s > 0). \end{aligned} \quad (134)$$

For the relevant values  $s = 0, 1, 2$ , it may also be noted that the Legendre polynomial can be expressed in terms of the ultraspherical polynomial according to

$$\begin{aligned} G_l^{1/2}(y) &= P_l(y) \\ \frac{d}{dy} G_{l+1}^{-1/2}(y) &= -P_l(y) \\ (l + 2)(l - 1) G_{l+2}^{-3/2}(y) + 2y \frac{d}{dy} G_{l+2}^{-3/2}(y) &= -(1 - y^2) P_l(y). \end{aligned} \quad (135)$$